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Determination of limit cycles for two-dimensional dynamical systems

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We consider in this paper two-dimensional dynamical systems of the form $\dot{x}=P(x,y),\dot{y}=Q(x,y)$, where P and Q are analytic functions. We introduce a method for finding the limit cycles of the system. This method consists of searching for a power series solution of the equation $P(\partial V/\partial x)+Q(\partial V/\partial y)=[(\partial P/\partial x)+(\partial Q/\partial y)]V$. The limit cycles are determined from the condition $V(x,y)=0$.

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I. INTRODUCTION

We consider two-dimensional dynamical systems of the form

$$\dot{x}=P(x,y),\quad \dot{y}=Q(x,y)\ , \tag{1.1}$$

where P and Q are two real functions of the real variables x and y, analytic in the whole plane (usually polynomials), and the overdots denote a time derivative. Such types of dynamical systems appear very often within several branches of science, such as biology, chemistry, astrophysics, mechanics, electronic, fluid mechanics, etc. Even for the case $n=2$, the field of applications of (1.1) is very extensive [1].

For a given system of type (1.1), it is a very difficult problem to determine the number of limit cycles and their configurations in phase space (a limit cycle is an isolated periodic solution) [2]. If P and Q are polynomials of degree n, a question of interest is to find the maximum possible number of limit cycles for a given value of n. Each for the case $n=2$, this problem has not been solved. Regarding the problem of the distribution of limit cycles in the x-y plane, there has also been very little progress made. Some results in this direction have been obtained only for special cases [3–5]. The determination of an equation which gives the limit cycles in the x-y plane is another problem where almost nothing is known.

In this paper we present a method for determining limit cycles of (1.1) that enclose a critical point P_0 of node or focus type, having the following properties: it is not degenerated and the ratio of the two eigenvalues of the linear part of (1.1) associated to this point is not a rational number. The method is based on the following results: if we consider the partial differential equation

$$P\frac{\partial V}{\partial x}+Q\frac{\partial V}{\partial y}=\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right]V \tag{1.2}$$

there exists a unique convergent power series solution of (1.2) in some region D containing P_0. This solution satisfies

$$V(x,y)=0 \tag{1.3}$$

on any limit cycle contained in D and enclosing P_0.

This paper is organized as follows. In Sec. II we present the method for determining limit cycles and we prove the result (1.3). In Sec. III we prove the existence and uniqueness of an analytic solution of (1.2) in some neighborhood of P_0. In Sec. IV we consider a case that presents limit cycles and for which the function $V(x,y)$ can be explicitly calculated; in this case, V is analytic in the whole plane and then all the limit cycles are obtained from $V(x,y)=0$. In Sec. V we analyze two particular systems for which it is known that a unique limit cycle exists; we calculate for these cases the function $V(x,y)$ as a power series x and y and study the localization of limit cycles from Eq. (1.3), at different orders of the expansion. Finally, in Sec. VI we present our conclusions.

II. LIMIT CYCLES AND THE RECIPROCAL OF THE INTEGRATING FACTOR

Let $u_1(t)$ and $u_2(t)$ be a particular solution of (1.1), i.e.,

$$\dot{u}_1=P(u_1,u_2),\quad \dot{u}_2=Q(u_1,u_2)\ . \tag{2.1}$$

The variational equations associated with $u_1(t),u_2(t)$ are obtained by linearizing (1.1) in a neighborhood of this solution. It is accomplished by writing

$$x=u_1+\epsilon v_1,\quad y=u_2+\epsilon v_2\ , \tag{2.2}$$

where ϵ is a small constant parameter.

Substituting (2.2) in (2.1) and keeping only the first order in ϵ, we obtain

$$\begin{aligned}\dot{v}_1&=v_1\frac{\partial P}{\partial x}(u_1,u_2)+v_2\frac{\partial P}{\partial y}(u_1,u_2)\ ,\\ \dot{v}_2&=v_1\frac{\partial Q}{\partial x}(u_1,u_2)+v_2\frac{\partial Q}{\partial y}(u_1,u_2)\ .\end{aligned} \tag{2.3}$$

These are the so-called variation equations associated with the particular solution (u_1,u_2). They are a nonautonomous system of two first order linear differential equations. When a solution of (2.3) is replaced in (2.2) we obtain a solution of (1.1), up to the first order in ϵ. When the particular solution (u_1,u_2) is known, (2.3) is usually employed for studying the behavior of solutions of (1.1) in the neighborhood of (u_1,u_2). This study permits one to analyze the stability properties of the particular solution.

Here, we employ the variational equations (2.3) in a different way. Equations (2.3) always have the special solution

$$v_1=P(u_1,u_2), \quad v_2=Q(u_1,u_2) . \tag{2.4}$$

This solution defines a perturbation of (u_1,u_2) in the tangent direction to this trajectory and it depends on t, only through the functions $u_1(t)$ and $u_2(t)$. The curve that represents this perturbed solution is the same as the curve associated with (u_1,u_2). A point of the curve represented by the particular solution (u_1,u_2) is transformed, by the perturbation (2.4), in another point of the same curve (obviously up to the first order in ϵ). As it is well known [5], (2.4) is the only perturbation that has this property.

More generally, a perturbation having the form

$$v_1=v_1(u_1,u_2), \quad v_2=v_2(u_1,u_2) , \tag{2.5}$$

where $v_1(x,y)$ and $v_2(x,y)$ are also two analytic functions as in (2.4), must satisfy the following system of two first order linear partial differential equations:

$$\begin{aligned} P\frac{\partial v_1}{\partial x}+Q\frac{\partial v_1}{\partial y}&=v_1\frac{\partial P}{\partial x}+v_2\frac{\partial P}{\partial y} , \\ P\frac{\partial v_2}{\partial x}+Q\frac{\partial v_2}{\partial y}&=v_1\frac{\partial Q}{\partial x}+v_2\frac{\partial Q}{\partial y} . \end{aligned} \tag{2.6}$$

These equations are obtained by replacing (2.5) in (2.3). Let us choose a particular solution $(u_1(t)=u_{1cl}(t),u_2(t)=u_{2cl}(t))$ representing a limit cycle of (1.1). The perturbed solution $[u_{1cl}(t)+\epsilon v_1(u_{1cl}(t), u_{2cl}(t)), u_{2cl}(t)+\epsilon v_2(u_{1cl}(t), u_{2cl}(t))]$ is a periodic function of t. But a limit cycle is an isolated periodic solution of (1.1). Therefore, this perturbed solution must represent the same curve as the unperturbed one (i.e., the limit cycle itself). The only infinitesimal perturbation, up to a constant factor α, that leaves invariant a trajectory $(u_1(t),u_2(t))$ is $(\epsilon v_1=\epsilon P(u_1,u_2), \epsilon v_2=\epsilon Q(u_1,u_2))$. As a consequence, an analytic solution (v_1,v_2) of (2.6), calculated on the limit cycle, must satisfy

$$\begin{aligned} v_1(u_{1cl}(t),u_{2cl}(t))&=\alpha P(u_{1cl}(t),u_{2cl}(t)) , \\ v_2(u_{1cl}(t),u_{2cl}(t))&=\alpha Q(u_{1cl}(t),u_{2cl}(t)) . \end{aligned} \tag{2.7}$$

Let us now introduce the function

$$V=Pv_2-Qv_1 . \tag{2.8}$$

Then, taking into account (2.7) we obtain

$$V(u_{1cl},u_{2cl})=0 . \tag{2.9}$$

Therefore all the limit cycles of (1.1) must satisfy Eqs. (2.9). Moreover, using (2.6), we can see that the function V satisfies the following partial differential equation:

$$P\frac{\partial V}{\partial x}+Q\frac{\partial V}{\partial y}=\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right]V . \tag{2.10}$$

Equations (2.9) and (2.10) represent the fundamental results of this paper. We can enounce these results as follows: let γ be a limit cycle of the system (1.1). Let V be an analytic solution of (2.10) in some region D of $\mathbb{R}^2$. If γ is contained in D, then V is zero on γ.

Let us remark that, for $V\neq 0$, $M=1/V$ satisfies the equation

$$P\frac{\partial M}{\partial x}+Q\frac{\partial M}{\partial y}=-\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right]M . \tag{2.11}$$

Consequently, V is the reciprocal of an integrating factor of the equation $Q\,dx-P\,dy=0$.

III. SEARCH FOR A POWER SERIES SOLUTION OF EQ. (2.10)

We assume that (1.1) has one or more critical points. We choose one of them and we locate it at the origin. To determine limit cycles that enclose this point we seek for a power series solution of (2.10) in a neighborhood of the origin.

A. Existence and uniqueness of the formal series solution of (2.10)

To simplify the calculations we assume that the linear part of P and Q has been transformed by a linear change of variables to its diagonal form:

$$\begin{aligned} P(x,y)&=p_1x+p_3x^2+p_4xy+p_5y^2+\cdots , \\ Q(x,y)&=q_2y+q_3x^2+q_4xy+q_5y^2+\cdots , \end{aligned} \tag{3.1}$$

where p_1 and q_2 are the eigenvalues of the linear part of (1.1) (here P and Q can be complex polynomials).

Equation (2.10) evaluated at the critical point (0,0) gives

$$(p_1+q_2)V(0,0)=0 .$$

If $p_1+q_2\neq 0$ then

$$V(0,0)=0 .$$

Let us point out that, for a simple focus or a node type critical point, p_1+q_2 is always nonzero. Consequently, the function V always vanishes on these two types of critical points. Let us suppose now that $p_1q_2\neq 0$ and $p_1/q_2\notin\mathbb{Q}$. Taking partial derivatives of (2.10) at the origin we obtain

$$p_1\frac{\partial V}{\partial y}(0,0)=0, \quad q_2\frac{\partial V}{\partial x}(0,0)=0 ,$$

then

$$\frac{\partial V}{\partial x}(0,0)=0, \quad \frac{\partial V}{\partial y}(0,0)=0 .$$

Therefore, the expansion of V in powers of x and y can be written as

$$V(x,y)=c_{20}x^2+c_{11}xy+c_{02}y^2+\cdots+\sum_{k=0}^{n}c_{n-k,k}x^{n-k}y^k+\cdots . \tag{3.2}$$

Replacing (3.2) in (2.10) we find $c_{20}=c_{02}=0$ and c_{11} (which will be called c in the following) arbitrarily valued. The calculation of the coefficients of terms of degree 3 yields the following 4×4 linear system of equations:

$$\begin{bmatrix} 2p_1-q_2 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & q_2 & 0 \\ 0 & 0 & 0 & 2q_2-p_1 \end{bmatrix} \begin{bmatrix} c_{30} \\ c_{21} \\ c_{12} \\ c_{03} \end{bmatrix} = c \begin{bmatrix} -q_3 \\ p_3 \\ q_5 \\ -p_5 \end{bmatrix} . \quad (3.3)$$

As a consequence of the hypothesis on the eigenvalues, this system of equations has a unique solution, each coefficient c_{ij} being proportional to c. This result can be generalized to all orders. At order n, the $(n+1)$ coefficients c_{ij} of the homogeneous polynomial of degree n are also calculated from a diagonal linear system of equations. The determinant Δ_n of this system is a function only of p_1 and q_2 and is given by

$$\Delta_n = \prod_{k=0}^{n} [(n-1-k)p_1+(k-1)q_2] . \quad (3.4)$$

The coefficients $c_{n-k,k}$ with $0 \le k \le n$ are determined in a unique way and can be written as

$$c_{n-k,k} = \frac{c f_{n,k}(p_i,q_i)}{(n-1-k)p_1+(k-1)q_2} , \quad (3.5)$$

where $f_{n,k}$ is a polynomial in the variables p_i and q_i.

Then, under the above hypothesis, Eq. (2.10) has a unique formal series solution in powers of x and y, up to a multiplicative constant factor. The conditions we have imposed in order to ensure the existence and uniqueness of the formal series solution of (2.10) are sufficient but not necessary. Anyway, the cases that are excluded are not generic. An arbitrarily small perturbation of the system can always be performed in a nongeneric case in such a way that the new system will satisfy these conditions.

B. Local existence of a convergent power series solution of (2.10) in a neighborhood of node and focus type critical points

The determination of the domain of convergence of the formal series (3.2) is a very difficult problem and it is not possible to establish general results about it, even for the simplest case where P and Q are second degree polynomials. Nevertheless, it is easy to prove the local existence of a convergent power series solution of (2.10). Here we use the initial form of the system (1.1) where P and Q are two real analytic functions [the linear part of (1.1) is not necessarily of a diagonal form].

1. *Focus type critical point*

Let the origin be a critical point of focus type of (1.1), with eigenvalues $\lambda=\alpha\pm i\beta$ ($\alpha\neq 0$, $\beta\neq 0$). Then, there exists a neighborhood of the origin in which equation (2.10) has a convergent power series solution. For the proof we employ a result due to Poincaré [6]: under the above hypothesis, the equations

$$\begin{aligned} P\frac{\partial f_1}{\partial x}+Q\frac{\partial f_1}{\partial y} &= \alpha f_1-\beta f_2 , \\ P\frac{\partial f_2}{\partial x}+Q\frac{\partial f_2}{\partial y} &= \alpha f_2+\beta f_1 \end{aligned} \quad (3.6)$$

have at least a nonzero convergent power series solution in some neighborhood of the origin, and this solution satisfies $f_1(0,0)=f_2(0,0)=0$ and $J(0,0)\neq 0$, where

$$J(x,y)=\left[\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y}-\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\right](x,y) .$$

The functions f_1 and f_2 represent the change of variables that linearizes (1.1) in a neighborhood of the critical point. In this neighborhood, under the change of variables

$$X=f_1(x,y), \quad Y=f_2(x,y) ,$$

the system (1.1) is analytically equivalent to the linear system

$$\dot{X}=\alpha X-\beta Y, \quad \dot{Y}=\alpha Y+\beta X .$$

This linear system has the first integral

$$I=-\beta \ln(X^2+Y^2)+2\alpha \arctan\frac{Y}{X} .$$

Therefore, we have also a first integral for (1.1) given by

$$I=-\beta \ln[f_1^2(x,y)+f_2^2(x,y)] + 2\alpha \arctan\frac{f_2(x,y)}{f_1(x,y)} .$$

From this constant of motion we find a reciprocal of an integrating factor given by

$$V(x,y)=\frac{f_1^2(x,y)+f_2^2(x,y)}{J(x,y)} .$$

It is easy to verify, by a direct calculation, that this function $V(x,y)$ satisfies (2.10). In addition, as $J(0,0)\neq 0$, $V(x,y)$ is analytic in some neighborhood of the origin.

2. *Node type critical point*

Let the origin be a critical point of (1.1) of node type, with real eigenvalues λ_1 and λ_2, such that $\lambda_1\lambda_2\neq 0$ and $\lambda_1/\lambda_2\notin\mathbb{Q}$. Then, there exists a neighborhood of the origin in which Eq. (2.10) has a convergent power series solution. For the proof we employ again Poincaré's result [6]: under the above hypothesis, the equations

$$P\frac{\partial f_1}{\partial x}+Q\frac{\partial f_1}{\partial y}=\lambda_1 f_1, \quad P\frac{\partial f_2}{\partial x}+Q\frac{\partial f_2}{\partial y}=\lambda_2 f_2 \quad (3.7)$$

have at least a nonzero convergent power series solution in some neighborhood of the origin, and this solution satisfies $f_1(0,0)=f_2(0,0)=0$ and $J(0,0)\neq 0$. In this case, the system (1.1) has the first integral

$$I=\lambda_2 \ln|f_1(x,y)|-\lambda_1 \ln|f_2(x,y)| .$$

From this expression we deduce a reciprocal of an integrating factor given by

$$V(x,y)=\frac{f_1(x,y)f_2(x,y)}{J(x,y)} .$$

As in Sec. III B 1 it is easy to verify that this function $V(x,y)$ satisfies (2.10). In addition, as $J(0,0)\neq 0$, $V(x,y)$ is analytic in some neighborhood of the origin.

IV. A CASE THAT PRESENTS LIMIT CYCLES AND FOR WHICH $V(x,y)$ CAN BE EXPLICITLY CALCULATED

Let us consider the system

$$\dot{x}=xg[x^2+y^2]-y, \quad \dot{y}=yg[x^2+y^2]+x , \tag{4.1}$$

where $g[u]$ is a polynomial function of u and the brackets indicate the argument of the function. Writing this system in polar coordinates, it is easy to show that it presents n limit cycles, where n is the number of positive real roots of g: $R_1^2,R_2^2,\dots,R_n^2$. These limit cycles are circles of radius R_i, centered at the origin, which is the only critical point of (4.1). Therefore, all the limit cycles of (4.1) are determined from the equation $g[x^2+y^2]=0$. If $a_0\neq 0$ is the zero degree term of g, the linear part of (4.1) at the origin is

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}a_0 & -1\\ 1 & a_0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} .$$

The eigenvalues λ_1 and λ_2 of this linear part are the roots of $1+(a_0-\lambda)^2$ and then they satisfy conditions of Sec. III A. We have proved in this section that there exists a unique formal series solutions of (2.10). For the system (4.1) this equation takes the form

$$2[g+(x^2+y^2)g']V=(xg-y)\frac{\partial V}{\partial x}+(yg+x)\frac{\partial V}{\partial y} , \tag{4.2}$$

where g' indicates the derivative of g with respect to u. It can be easily verified that

$$V(x,y)=(x^2+y^2)g[x^2+y^2] \tag{4.3}$$

is a solution of (4.2). This solution is analytic in the whole plane and is, therefore, the unique solution of this type. We see that the equation $W(x,y)=0$ contains all the limit cycles and the critical point of (4.1). If $g[u]$ has no positive real roots, then (4.1) has no limit cycles. In this case, the condition $V(x,y)=0$ determines only a trajectory of (4.1): the critical point. Let us consider the particular case

$$g[u]=(1-u)\left[1-\frac{u}{4}\right]$$

for which the system (4.1) has two limit cycles, the circles of radius $R_1=1$ and $R_2=2$. For this system a solution of (3.6) is given by

$$f_1(x,y)=\frac{x|x^2+y^2-4|^{1/6}}{|x^2+y^2-1|^{2/3}} ,$$

$$f_2(x,y)=\frac{y|x^2+y^2-4|^{1/6}}{|x^2+y^2-1|^{2/3}} .$$

Let us briefly explain the method which leads to this result. As the system (4.1) satisfies the conditions of Sec. III A, it has a first integral of the form

$$I=-\beta\ln[f_1^2(x,y)+f_2^2(x,y)]+2\alpha\,\mathrm{arctg}\frac{f_2(x,y)}{f_1(x,y)} .$$

Moreover, this system is completely integrable, as can be seen by writing it in polar coordinates. In consequence, an explicit expression of the first integral can be given and the functions f_1,f_2 can be determined. The common domain of analyticity $D(f_1,f_2)$ of f_1 and f_2 is the interior of the limit cycle of radius $R_1=1$, whereas $V(x,y)=(f_1^2+f_2^2)/J=(x^2+y^2)g[x^2+y^2]$ is analytic in the whole plane.

For the general case (1.1), $D(f_1,f_2)$ cannot contain any limit cycles, because, in this domain, the system is analytically equivalent to its linear part. In consequence, if the domain of convergence $D(V)$ of V is not greater than $D(f_1,f_2)$, the function V is not useful for detecting limit cycles. But, from the example considered above, we can hope that, in the general case, $D(V)$ is greater than $D(f_1,f_2)$. Unfortunately, it is difficult to establish a relation between $D(V)$ and $D(f_1,f_2)$.

V. POWER SERIES SOLUTION OF EQ. (2.10) FOR TWO SYSTEMS THAT PRESENT A LIMIT CYCLE

In this section we study two representative examples among all the systems that we have analyzed using our method. We seek $V(x,y)$ as a power series in x and y. Let us write $V(x,y)$ as

$$V(x,y)=\sum_{n=0}^{\infty} v_n(x,y) , \tag{5.1}$$

where the homogeneous polynomial of degree n, $v_n(x,y)$, is given by

$$v_n(x,y)=\sum_{k=0}^{n} c_{n-k,k}x^{n-k}y^k . \tag{5.2}$$

The truncated sum at order N is

$$V_N(x,y)=\sum_{n=2}^{N} v_n(x,y) . \tag{5.3}$$

For the two examples considered in this section we have studied the curves obtained from the conditions $V_N(x,y)=0$, with increasing values of N. We have compared these curves with the numerical results obtained with the Runge-Kutta method. In particular, we have compared the coordinates of the intersection points of the limit cycle with the coordinate axes.

The first example that we consider is the well known van der Pol equation for $\epsilon=1$:

$$\dot{x}=y, \quad \dot{y}=-x+\epsilon(1-x^2)y . \tag{5.4}$$

This system has a unique limit cycle that encloses the only critical point, located at the origin. The eigenvalues of the linear part of (5.4) at this critical point are $(1\pm i\sqrt{3})$. Therefore, there exists a unique formal series solution of (2.10). We have calculated the coefficient

TABLE I. Intersections of curves $V_N(x,y)=0$ with x and y axes for the van der Pol equation (system 5.4). When x_1, x_2 are complex numbers, the curve is open.

Order N	Intersections with y axis y_1,y_2	Intersections with x axis x_1,x_2
8	± 2.150	± 1.243
10	± 2.274	$\pm(1.247\pm i0.380)$
12	± 2.203	± 1.343
14	± 2.194	$\pm(1.364\pm i0.293)$
16	± 2.185	± 1.429
18	± 2.179	$\pm(1.454\pm i0.239)$
20	± 2.175	± 1.504
22	± 2.173	$\pm(1.523\pm i0.204)$
24	± 2.171	± 1.570
26	± 2.170	$\pm(1.595\pm i0.178)$
28	± 2.16944	± 1.629
30	± 2.16896	$\pm(1.653\pm i0.159)$
32	± 2.16867	± 1.683
34	± 2.16850	$\pm(1.706\pm i0.143)$
36	± 2.16844	± 1.732

$c_{n-k,k}$ of (5.2) employing the MATHEMATICA computer algebra system. In this case, the series (5.1) contains only homogeneous polynomials of even degree; this is a consequence of the invariance of (5.4) under the transformation $x \to -x, y \to -y$.

We found that, from $N=8$, the curves $V_N(x,y)=0$ are alternatively closed for the orders $N=4j$ and open for the orders $N=4j+2$, with $j=1,2,3,\ldots$ For $N=4j+2$ the equation $V_N(x,0)=0$ has only complex roots. Among them, there are two pairs of complex conjugates roots which have a real part very near the values that have been found for the precedent order ($N=4j$). The imaginary part of these two pairs of roots is small and decreases as N increases (see Table I).

Table I gives the values of the intersection points of the curves $V_N(x,y)=0$ with the coordinate axes, for $8 \le N \le 36$. The intersections with the y axis are rapidly very near the numerical value $|y|=2.17$, obtained with the Runge-Kutta method. The intersections with the x-axis approach more slowly, but in a monotonous way, to the exact value $|x|=2$. The curves $V_N(x,y)=0$ are closed, for $N=4j$, as early as $N=8$. For the value $N=8$, the curve is shown in Fig. 1; it is very surprising to see that it reproduces some of the qualitative properties of the limit cycle of the van der Pol equation. The curves obtained from greater values of $N=4j$ regularly approach to the numerical trajectory, as it is shown in Figs. 2 and 3, for $N=20$ and $N=32$, respectively. The analysis of the coefficients of the series (5.1) for the van der Pol equation leads us to believe that the limit cycle is completely contained in the interior of the domain of convergence of (5.1).

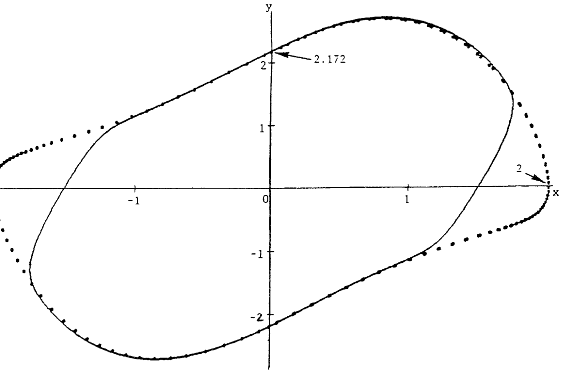

FIG. 2. Curve $V_{20}(x,y)=0$ for the van der Pol equation. The dotted line represents the numerical curve obtained by the Runge-Kutta method.

Our second example is given by the system

$$\dot{x}=-y+\tfrac{1}{10}x+x^2, \quad \dot{y}=x(1+x+y)\ . \tag{5.5}$$

It is proved in [4] that this system has a unique limit cycle that encloses the only critical point, located at the origin. The two eigenvalues of the linear part of (5.5) at the critical point are complex conjugate numbers with nonzero real part. Therefore, the formal series solution of (2.10) is unique for this system. Here, the series (5.1)

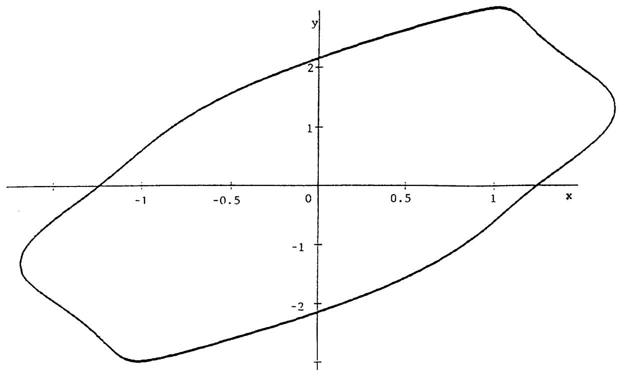

FIG. 1. Curve $V_8(x,y)=0$ for the van der Pol equation (system 5.4).

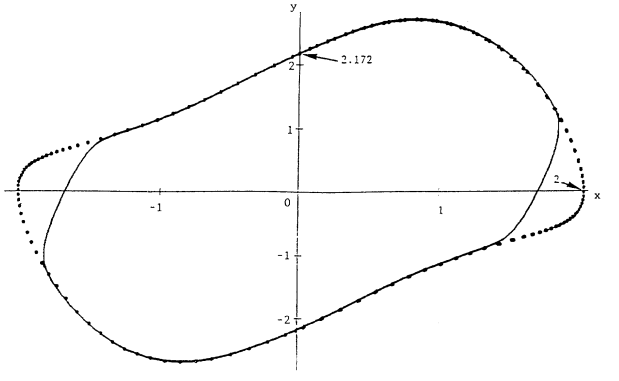

FIG. 3. Curve $V_{32}(x,y)=0$ for the van der Pol equation. The dotted line represents the numerical curve obtained by the Runge-Kutta method.

TABLE II. Intersections of curves $V_N(x,y)=0$ with x and y axes for system 5.5. When y_2 is a complex number, the curve is open.

Order N	Intersections with x axis x_1	x_2	Intersections with y axis y_1	y_2
4	0.149	−0.327	−0.309	0.443
6	0.327	−0.373	−0.263	0.456
7	0.321	−0.414	−0.258	Complex
8	0.318	−0.386	−0.256	0.458
9	0.317	−0.402	−0.2546	Complex
10	0.3159	−0.391	−0.2539	0.456
11	0.3156	−0.398	−0.2535	Complex
12	0.31539	−0.393	−0.2533	0.453
13	0.31528	−0.396	−0.2532	$0.457+i0.096$
14	0.31523	−0.3938	−0.2531	0.451
15	0.315207	−0.3951	−0.25307	$0.453+i0.084$
16	0.315196	−0.3944	−0.25305	0.451
17	0.315192	−0.3947	−0.25304	$0.455+i0.074$
18	0.315192	−0.3947	−0.253054	0.458
19	0.315194	−0.3945	−0.253033	$0.474+i0.065$
20	0.315196	−0.3948	−0.253032	$0.496+i0.064$
21	0.315199	−0.3944	−0.253033	0.452
22	0.315200	−0.3948	−0.253034	$0.437+i0.058$
23	0.315201	−0.3944	−0.253036	0.428

contains homogeneous polynomials of even and odd orders. Between $N=4$ and $N=19$ the curves $V_N(x,y)=0$ are closed for N even and open for N odd. From $N=20$ and above the situation is reversed: the curves are open for N even and closed for N odd.

Table II gives the intersections of these curves with the coordinates axes for $4 \leq N \leq 23$. We see that the intersections with the x axis are rapidly very near to the numerical values $x_1=-0.394$ and $x_2=0.315$, obtained with the Runge-Kutta method. With respect to the intersections with the y axis, they are also very satisfactory for the negative intersections, with a sequence that rapidly approaches the numerical value $y_1=-0.253$. The convergence of the sequence that gives the positive intersections, up to the order $N=23$, is more doubtful. The bottom part of the curves $V_N(x,y)=0$ approaches very well the numerical curve, while the upper part does not.

In Fig. 4 we exhibit the curve $V_4(x,y)=0$, which is the closed curve of the smallest order. As in the case of the curve $V_8(x,y)=0$ for the van der Pol equation, this curve contains, already for $N=4$, some qualitative aspects of the limit cycle of (5.5).

The analysis of the coefficients of (5.1) and Figs. 5 and 6 for this example, seem to indicate that the domain of convergence of the series (5.1) contains the bottom part of the curve, but, perhaps, does not contain the upper one.

If we are only interested in the problem of the existence of the limit cycle and its approximate localization in phase space, it seems to be sufficient to study the

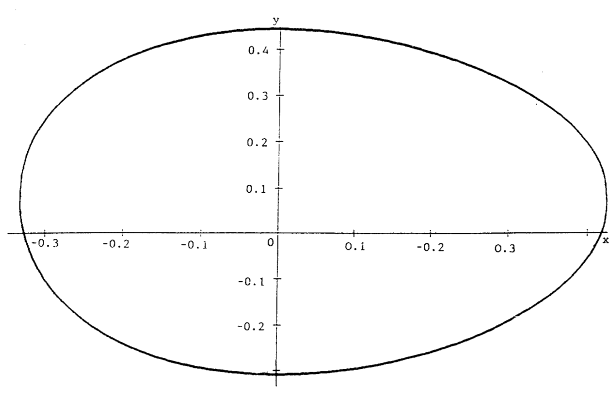

FIG. 4. Curve $V_4(x,y)=0$ for system (5.5).

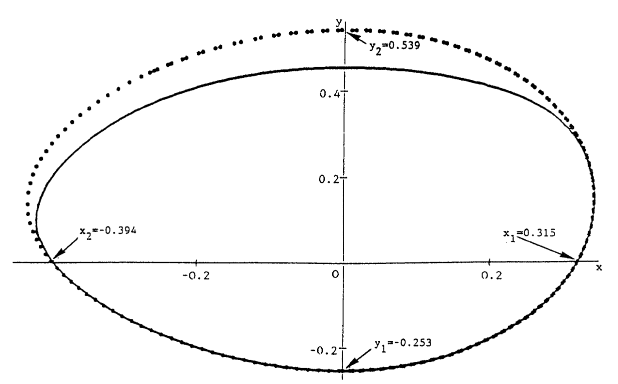

FIG. 5. Curve $V_{12}(x,y)=0$ for system (5.5). The dotted line represents the numerical curve obtained by the Runge-Kutta method.

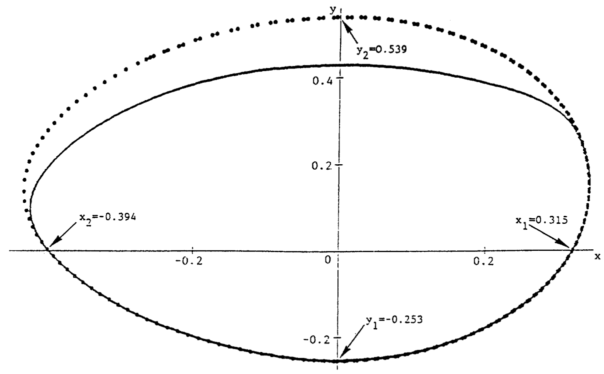

FIG. 6. Curve $V_{23}(x,y)=0$ for system (5.5). The dotted line represents the numerical curve obtained by the Runge-Kutta method.

curves $V_N(x,y)=0$ for small values of N. The same type of behavior has been found in all the examples that we have studied with the MATHEMATICA computer algebra system.

We have also studied a case that does not present limit cycles:

$$\dot{x}=\alpha x-y+(\alpha+1)x^2-xy, \quad \dot{y}=x+x^2 \quad (\text{with } \alpha=-\tfrac{1}{2}) . \tag{5.6}$$

It has been shown in [4] that the system (5.6) does not have limit cycles for $\alpha\in[-1,0]$. We calculated the coefficients of (5.1) for the system (5.6), up to order $N=12$. No closed curve has been found up to this order.

From the examples analyzed in this section we find that the curves defined by the conditions $V_N(x,y)=0$ give, even for small values of N, a good qualitative information on the existence of limit cycles and their localization in phase space.

VI. CONCLUSIONS

In this paper we have studied the limit cycles of two-dimensional analytic systems of the form (1.1). The fundamental result that we have obtained is the following: the partial differential equation

$$P\frac{\partial V}{\partial x}+Q\frac{\partial V}{\partial y}=\left[\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right]V$$

has a unique convergent power series solution in a region D containing a nondegenerate critical point P_0 of node or focus type, when the ratio of the two eigenvalues associated with P_0 is not a rational number. *This solution vanishes on any limit cycle, contained in D and enclosing* P_0. In Sec. IV we furnished examples for which all limit cycles enclosing a given critical point are contained in the region of convergence of V. In Sec. V we employed this result to localize limit cycles for two particular systems, by explicitly calculating the formal series solution of the partial differential equation written above. For the van der Pol equation, the limit cycle seems to be completely contained in the region of convergence of V. For the example (5.5), the region of convergence of V seems to contain only a part of the limit cycle. Obviously, no definite answer can be given about convergence properties by using truncated series expansions.

The more important open question concerning this work is, for the general case, how many limit cycles enclosing a critical point are contained in the convergence region of V. Another interesting problem to be considered is the study of systems that have several free parameters. In this case, the conditions $V_N(x,y)=0$ can be analyzed as a function of these parameters and employed to detect the possible bifurcations that occur as the parameters are varied.

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[1] J. W. Reyn, Nieuw. Arch Wisk. **5**, 107 (1987).
[2] D. Delamotte, Phys. Rev. Lett. **70**, 3361 (1993); D. Poland, Phys. Rev. E **49**, 157 (1994).
[3] N. G. Lloyd, in *New Directions in Dynamical Systems*, London Math. Soc. Lecture Note Séries No. 127, edited by T. Bedford and J. Swift (Cambridge University Press, Cambridge, England, 1988), p. 192.
[4] Ye Yan-Qian *et al.*, *Theory of Limit Cycles*, translations of mathematical monographs, Vol. 66 (American Mathematical Society, Providence, 1986).
[5] V. V. Nemysky and V. V. Stepanov, *Qualitative Theory of Differential Equations* (Princeton, University Press, Princeton, New Jersey, 1960); V. I. Arnold and Yu. S. Il'Yashenko, in *Dynamical Systems*, Encyclopaedia of Mathematical Sciences Vol. 1 (Springer-Verlag, Berlin, 1988); L. Perko, *Differential Equations and Dynamical Systems*, Texts in Applied Mathematics No. 7 (Springer-Verlag, New York, 1991).
[6] H. Poincaré, *Thèse et "Mémoire sur les courbes définies par une équation differentielle,"* Oeuvres (1880–1890) (Gauthier-Villars, Paris, 1929).